Exercise sheet #7

Problem 1. A very long cylinder, of radius a, carries a uniform polarization $\mathbf{P}$ perpendicular to its axis. Find the electric field inside the cylinder. Show that the field outside the cylinder can be expressed in the form

$$\mathbf{E}(\mathbf{r})=\frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}}-\mathbf{P}]$$

[Careful: I said "uniform," not "radial"!]

Solution: Think of it as two cylinders of opposite uniform charge density $\pm\rho$. Inside, the field at a distance s from the axis of a uniformly charge cylinder is given by Gauss's law: $E2\pi s\ell=\frac{1}{\epsilon_0}\rho\pi s^2\ell \Rightarrow \mathbf{E}=(\rho/2\epsilon_0)\,\mathbf{s}$. For two such cylinders, one plus and one minus, the net field (inside) is $\mathbf{E}=\mathbf{E}_++\mathbf{E}_-=(\rho/2\epsilon_0)\left(\mathbf{s}_+-\mathbf{s}_-\right)$. But $\mathbf{s}_+-\mathbf{s}_-=-\mathbf{d}$, so $\mathbf{E}=-\rho\mathbf{d}/\left(2\epsilon_0\right)$, where $\mathbf{d}$ is the vector from the negative axis to positive axis. In this case the total dipole moment of a chunk of length ℓ is $\mathbf{P}\left(\pi a^2\ell\right)=\left(\rho\pi a^2\ell\right)\mathbf{d}$. So $\rho\mathbf{d}=\mathbf{P}$, and $\boxed{\mathbf{E}=-\mathbf{P}/\left(2\epsilon_0\right)}$, for $s<a$.

Outside, Gauss's law gives $E2\pi s\ell=\frac{1}{\epsilon_0}\rho\pi a^2\ell \Rightarrow \mathbf{E}=\frac{\rho a^2}{2\epsilon_0}\frac{\hat{\mathbf{s}}}{s}$, for one cylinder. For the combination, $\mathbf{E}=\mathbf{E}_++\mathbf{E}_-=\frac{\rho a^2}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+}-\frac{\hat{\mathbf{s}}_-}{s_-}\right)$, where

$$\begin{aligned}
\mathbf{s}_\pm &= \mathbf{s}\mp\frac{\mathbf{d}}{2}\\
\frac{\mathbf{s}_\pm}{s_\pm^2} &= \left(\mathbf{s}\mp\frac{\mathbf{d}}{2}\right)\left(s^2+\frac{d^2}{4}\mp\mathbf{s}\cdot\mathbf{d}\right)^{-1}\cong\frac{1}{s^2}\left(\mathbf{s}\mp\frac{\mathbf{d}}{2}\right)\left(1\mp\frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)^{-1}\cong\frac{1}{s^2}\left(\mathbf{s}\mp\frac{\mathbf{d}}{2}\right)\left(1\pm\frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)\\
&= \frac{1}{s^2}\left(\mathbf{s}\pm\mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2}\mp\frac{\mathbf{d}}{2}\right)\quad \text{(keeping only 1st order terms in } \mathbf{d}\text{).}
\end{aligned}$$

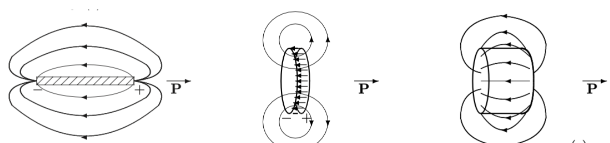
$$\left(\frac{\hat{\mathbf{s}}_+}{s_+}-\frac{\hat{\mathbf{s}}_-}{s_-}\right)=\frac{1}{s^2}\left[\left(\mathbf{s}+\mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2}-\frac{\mathbf{d}}{2}\right)-\left(\mathbf{s}-\mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2}+\frac{\mathbf{d}}{2}\right)\right]=\frac{1}{s^2}\left(2\frac{\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^2}-\mathbf{d}\right)$$

$$\boxed{\mathbf{E}(\mathbf{s})=\frac{a^2}{2\epsilon_0}\frac{1}{s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}}-\mathbf{P}]}\quad \text{for } s>a$$

□

Problem 2. A short cylinder, of radius a and length L, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L\gg a$, (ii) for $L\ll a$, and (iii) for $L\approx a$. [This is known as a bar electret; it is the electrical analog to a bar magnet. In practice, only very special materials-barium titanate is the most "familiar" example-will hold a permanent electric polarization. That's why you can't buy electrets at the toy store.]

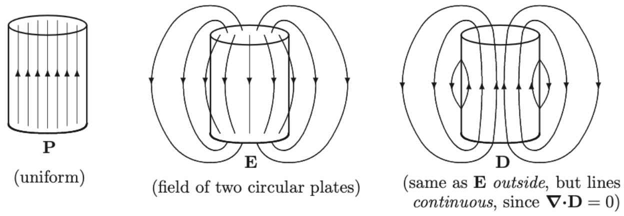
Solution: $\rho_b=0;\sigma_b=\mathbf{P}\cdot\hat{\mathbf{n}}=\pm P$ (plus sign at one end-the one $\mathbf{P}$ points toward; minus sign at the other-the one $\mathbf{P}$ points away from). (i) $L\gg a$. Then the ends look like point charges, and the whole thing is like a physical dipole, of length L and charge $P\pi a^2$. See Fig. (a). (ii) $L\ll a$. Then it's like a circular parallel-plate capacitor. Field is nearly uniform inside; nonuniform "fringing field" at the edges. See Fig. (b). (iii) $L\approx a$. See Fig. (c). □



(a) Like a dipole (b) Like a parallel-plate capacitor (c)

Problem 3. For the bar electret of the last problem, make three careful sketches: one of **P**, one of **E**, and one of **D**. Assume L is about $2a$. [Hint: **E** lines terminate on charges; **D** lines terminate on free charges.]

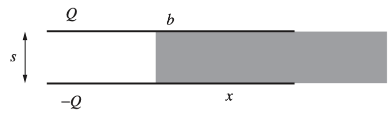
Solution: . □



Problem 4. A rectangular capacitor with side lengths a and b has separation s, with s much smaller than a and b. It is partially filled with a dielectric with dielectric constant κ. The overlap distance is x; see the figure below. The capacitor is isolated and has constant charge Q.

(a) What is the energy stored in the system? (Treat the capacitor like two capacitors in parallel.)

(b) What is the force on the dielectric? Does this force pull the dielectric into the capacitor or push it out?



Solution: (a) The equivalent capacitance of two capacitors in parallel is simply the sum of the capacitances. The capacitance of the part with the dielectric is κ times what it would be if there were vacuum there. So the total capacitance is given by

$$\begin{aligned} C = C_1 + C_2 &= \frac{\epsilon_0 A_1}{s} + \frac{\kappa\epsilon_0 A_2}{s} \\ &= \frac{\epsilon_0 a(b-x)}{s} + \frac{\kappa\epsilon_0 a x}{s} = \frac{\epsilon_0 a}{s}[b + (\kappa-1)x] \end{aligned}$$

The stored energy is then:

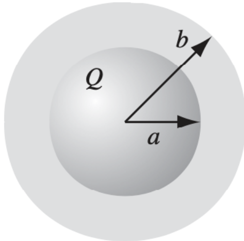
$$U = \frac{Q^2}{2C} = \frac{Q^2 s}{2\epsilon_0 a[b + (\kappa-1)x]}$$

Note that as x changes, the charge stays constant (by assumption), but the potential does not. So the $Q\phi/2$ and $C\phi^2/2$ forms of the energy aren't useful. (b) The force is:

$$F = -\frac{dU}{dx} = \frac{Q^2 s(\kappa-1)}{2\epsilon_0 a[b + (\kappa-1)x]^2}$$

The positive sign here means that the force points in the direction of increasing x. That is, the dielectric slab is pulled into the capacitor. But it's risky to trust this sign blindly. Physically, the force points in the direction of decreasing energy. And we see from the above expression for U that the energy decreases as x increases (because $\kappa > 1$). The force F is correctly zero if $\kappa = 1$, because in that case we don't actually have a dielectric. The $\kappa \to \infty$ limit corresponds to a conductor. In that case, both U and F are zero. Basically, all of the charge on the plates shifts to the overlap x region, and compensating charge gathers there in the dielectric, so in the end there is no field anywhere. Note that F decreases as x increases. □

Problem 5. A metal sphere of radius a carries a charge Q (See figure below). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).



Solution: To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$. We seem to be in a bind. What we *do* know is the free charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \qquad \text{for all points } r > a$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{0}$.) Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$:

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b \end{cases}$$

The potential at the center is therefore

$$\begin{aligned} V &= -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_b^a \left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_a^0 (0) dr \\ &= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right) \end{aligned}$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}$$

in the dielectric, and hence

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = 0$$

while

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \frac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface} \\ \frac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface} \end{cases}$$

Notice that the surface bound charge at a is negative ($\hat{\mathbf{n}}$ points outward with respect to the dielectric, which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $1/4\pi\epsilon_0\left(Q/r^2\right)\hat{\mathbf{r}}$ to $1/4\pi\epsilon\left(Q/r^2\right)\hat{\mathbf{r}}$. In this respect, a dielectric is rather like an imperfect conductor: on a conducting shell the induced surface charge would be such as to cancel the field of Q completely in the region $a < r < b$; the dielectric does the best it can, but the cancellation is only partial.

□